The Jacobian Conjecture is stably equivalent to the Dixmier Conjecture

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1 Introduction

The **Jacobian Conjecture** JC_n in dimension $n \geq 1$ asserts that *for any field* $\mathbf{k}$ *of characteristic zero any polynomial endomorphism* ϕ *of the* n*-dimensional affine space* $\mathbb{A}^n_{\mathbf{k}} = \mathsf{Spec}\,\mathbf{k}[x_1, \ldots, x_n]$ *over* $\mathbf{k}$*, with Jacobian* 1*:*

$$\det\left(\partial\phi^*(x_i)/\partial x_j\right)_{1\leq i,j\leq n} = 1$$

is an automorphism. Equivalently, one can say that ϕ preserves the standard top-degree differential form $dx_1 \wedge \cdots \wedge dx_n \in \Omega^n(\mathbb{A}^n_{\mathbf{k}})$.

The reference due to this well known problem and related questions can be found in [5], [3].

By the Lefschetz principle it is sufficient to consider the case $\mathbf{k} = \mathbb{C}$. Obviously, JC_n implies JC_m if $n > m$. We denote by JC_∞ the stable Jacobian conjecture, the conjunction of conjectures JC_n for all finite n. The conjecture JC_n is obviously true in the case $n = 1$, and it is open for $n \geq 2$.

The **Dixmier Conjecture** DC_n for integer $n \geq 1$ (see [4]) asserts that *for any field* $\mathbf{k}$ *of characteristic zero any endomorphism of the* n*-th Weyl algebra* $A_{n,\mathbf{k}}$ *over* $\mathbf{k}$ *is an automorphism.*

Here $A_{n,\mathbf{k}}$ is the associative unital algebra over $\mathbf{k}$ with $2n$ generators $y_1, \ldots, y_{2n}$ and relations

$$[y_i, y_j] = \omega_{ij}\,,$$

where $(\omega_{ij})_{1\leq i,j\leq 2n}$ is the following standard $2n \times 2n$ skew-symmetric matrix:

$$\omega_{ij} = \delta_{i,j+n} - \delta_{i+n,j}\ .$$

The algebra $A_{n,\mathbf{k}}$ coincides with the algebra $D(\mathbb{A}^n_{\mathbf{k}})$ of polynomial differential operators on $\mathbb{A}^n_{\mathbf{k}}$. For any i, $1 \le i \le n$ element y_i acts as the multiplication operator by the variable x_i, and element y_{n+i} acts by the differentiation $\partial/\partial x_i$. Again, it is sufficient to consider the case $\mathbf{k} = \mathbb{C}$. The conjecture DC_n implies DC_m for $n > m$, and we can consider the stable Dixmier conjecture DC_∞. The conjecture DC_n is open for any $n \ge 1$.

It is well-known that DC_n implies JC_n (in particular DC_∞ implies JC_∞) (see [5], [3]). The argument is very easy. Let $\phi : \mathbb{A}^n_{\mathbf{k}} \to \mathbb{A}^n_{\mathbf{k}}$ be a counterexample to JC_n. Then ϕ is a non-invertible étale map, and it induces a pullback homomorphism ϕ^*_{diff} of the algebra of differential operators on $\mathbb{A}^n_{\mathbf{k}}$. The endomorphism ϕ^*_{diff} of the Weyl algebra preserves the degree of differential operators. Restricting ϕ^*_{diff} to zero order differential operators, we obtain the usual pullback ϕ^* of functions on $\mathbb{A}^n_{\mathbf{k}}$. By our assertion it is not surjective, hence we obtain a counterexample to DC_n.

Our result is an opposite implication. Namely, we prove the following

Theorem 1 *Conjecture* JC_{2n} *implies* DC_n.

In particular, we obtain that the stable conjectures JC_∞ and DC_∞ are equivalent.

Remark 1 *A. van den Essen ([5], Theorem 10,4.2) proved a weaker result: the conjecture* JC_{2n} *implies the invertibility of any endomorphism of* $A_{n,\mathbf{k}} = D(\mathbb{A}^n_{\mathbf{k}})$ *preserving the filtration by the degrees of differential operators.*

For the convenience of the reader, and in order to make the text self-contained, we include in the paper proofs of several known results scattered in the literature. During the preparation of this paper we have learned from K. Adjamagbo about the preprint [10] where two key results concerning the Weyl algebra in finite characteristic were established (Propositions 2 and 4 from Section 4 in the present paper), see also a very recent preprint [2].

Remark 2 *The present paper is written in the standard language of algebraic geometry. It is possible (and reasonable for some minds) to use the model-theoretic language of non-standart analysis, instead of schéme-theoretic considerations. In particular, in the proofs of several results of our paper one can use the reduction modulo an infinitely large prime.*

Remark 3 *After this paper was written, we were told by Ken Goodearl about a paper "Endomorphsims of Weyl algebra and p-curvatures" (Osaka Journal of Mathematics Volume 42, Number 2 (June 2005)) by Yoshifumi Tsuchimoto which contains the proof of our main result. The proofs by Tsuchimoto and in the present paper are different (although there are many similarities), hence we think that it is reasonable to keep our paper on archive.*

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2 A reformulation of the Jacobian conjecture

For given integers $n \geq 2, d \geq 1$ we denote by $\mathrm{JE}_{n,d}$ an affine scheme of finite type over $\mathbb{Z}$ representing the following functor. For any commutative ring R the set $\mathrm{JE}_{n,d}(R)$ is the set of endomorphisms f of R-algebra $R[x_1, \dots, x_n]$ such that

- $\det \left(\partial f(x_i)/\partial x_j\right)_{1 \leq i,j \leq n} = 1 \in R[x_1, \dots, x_n]$,
- $\deg(f(x_i)) \leq d \quad \forall i,\, 1 \leq i \leq n\,.$

We say that f as above is an *endomorphism of degree* $\leq d$ (and with Jacobian 1). The ring of functions $\mathcal{O}(\mathrm{JE}_{n,d})$ is finitely generated, its generators are coefficients $c_{i,\alpha}$ which appear in the universal endomorphism over $\mathcal{O}(\mathrm{JE}_{n,d})$:

$$f_{univ}(x_i) = \sum_{\alpha:\, |\alpha| \leq d} c_{i,\alpha} x^\alpha$$

Here $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$ is a multi-index, $x^\alpha := \prod_{i=1}^n x_i^{\alpha_i}$, $|\alpha| := \sum_{i=1}^n \alpha_i\,.$

Similarly, for $n \geq 2, d \geq 1, d' \geq 1$ we denote by $\mathrm{JA}_{n,d,d'}$ an affine scheme of finite type over $\mathbb{Z}$ parameterizing pairs of endomorphisms (f, f') of n-dimensional affine space, with Jacobian 1, of degrees $\leq d$ and $\leq d'$ respectively, mutually inverse to each other: $f \circ f' = f' \circ f = \mathrm{Id}_{\mathbb{A}^n}$.

We have an obvious forgetting map $\mathrm{pr}^{(J)}_{n,d,d'} : \mathrm{JA}_{n,d,d'} \to \mathrm{JE}_{n,d},\ (f, f') \mapsto f$ which is an immersion (i.e. $\mathrm{JA}_{n,d,d'}$ is identified with a locally closed

subscheme of $\mathrm{JE}_{n,d}$). The Jacobian conjecture JC_n means that for any $d \geq 1$

$$\mathrm{JE}_{n,d} \times \mathsf{Spec}\, \mathbb{Q} = \bigcup_{d' \geq 1} \mathrm{pr}^{(J)}_{n,d,d'}(\mathrm{JA}_{n,d,d'} \times \mathsf{Spec}\, \mathbb{Q}) \ .$$

For given n, d the set $X_{d'} := (\mathrm{JA}_{n,d,d'} \times \mathsf{Spec}\, \mathbb{Q}) \subset \mathrm{JE}_{n,d} \times \mathsf{Spec}\, \mathbb{Q}$ is a constructible set. Therefore, we get an infinite growing chain of constructible subsets $X_1 \subset X_2 \subset \dots$ of the scheme of finite type $\mathrm{JE}_{n,d} \times \mathsf{Spec}\, \mathbb{Q}$ over $\mathbb{Q}$.

Let us assume JC_n and fix an integer $d \geq 1$. Then $\cup_{d' \geq 1} X_{d'} = X$ where $X := \mathrm{JE}_{n,d} \times \mathsf{Spec}\, \mathbb{Q}$. Then it follows from the standard properties of constructible sets (see [6], Corrollaire 1.9.8, Chapitre IV) that there exists an integer d' such that $X_{d'} = X$. Alternatively, one can use a result of O. Gabber (see [3], Theorem 1.2) which says that for an automorphism f of $\mathbf{k}[x_1, \dots, x_n]$ of degree $\leq d$ in the above sense ($\mathbf{k}$ is a field of any characteristic), the inverse map has the degree $\leq d^{n-1}$. Hence one can a priori set $d' = d^{n-1}$. Anyhow, the Jacobian conjecture can be rephrased as the equality $\mathrm{JA}_{n,d,d'}(\mathbb{C}) = \mathrm{JE}_{n,d}(\mathbb{C})$.

The following statement is obvious.

Lemma 1 *Let $\phi : A \to B$ be an immersion of schemes of finite type over $\mathbb{Z}$. Then ϕ induces a bijection between $A(\mathbb{C})$ and $B(\mathbb{C})$ if and only if there exists a finite set of primes S such that ϕ induces a bijection between $A(\mathbf{k})$ and $B(\mathbf{k})$ for any field $\mathbf{k}$ with $\mathrm{char}\, \mathbf{k} \notin S \cup \{0\}$.*

We apply it to the projection $\mathrm{pr}^{(J)}_{n,d,d'}$. The conclusion is that the Jacobian conjecture JC_n is equivalent to the following

Conjecture 1 *(JC_n in finite characteristic) For any $d \geq 1$ there exists $d' \geq 1$ and a finite set of primes S such that for any field $\mathbf{k}$ with $\mathrm{char}\, \mathbf{k} \notin S \cup \{0\}$ and any polynomial map $\phi : \mathbb{A}^n_{\mathbf{k}} \to \mathbb{A}^n_{\mathbf{k}}$ of degree $\leq d$ with Jacobian 1, the inverse map exists and has degree $\leq d'$.*

The equivalence of JC_n and the above conjecture in finite characteristic was first established by K. Adjamagbo in [1].

3 More about Weyl algebras

3.1 Weyl algebras over an arbitrary base

One can define an algebra $A_{n,R}$ for arbitrary commutative ring R exactly in the same way as for fields of characteristic zero. This algebra is free as a R-

module. It has a canonical basis consisting of monomials $(y_1^{\alpha_1} \dots y_{2n}^{\alpha_{2n}})_{(\alpha_1,\dots,\alpha_{2n})\in\mathbb{Z}^{2n}_{\geq 0}}$. Although for any R the algebra $A_{n,R}$ maps to the algebra $D(\mathbb{A}^n_R)$ of differential operators acting on $R[x_1, \dots, x_n]$, these two algebras are not isomorphic in general. For example, if R is an algebra over $\mathbb{Z}/p\mathbb{Z}$ for some prime p, then the operator $(d/dx_1)^p$ is zero.

We say that an endomorphism f of the algebra $A_{n,R}$ *has degree* $\leq d$ if the image $f(y_i)$ of any generator $y_i \in A_{n,R},\ \ i = 1, \dots, 2n$ is a linear combination with coefficients in R of the monomials of degree $\leq d$. In a manner completely parallel to the previous section, we can define schemes of finite type $\mathrm{DE}_{n,d}$, $\mathrm{DA}_{n,d,d'}$, and the projection $\mathrm{pr}^{(D)}_{n,d,d'}$. Also, we can make a reformulation of the Dixmier conjecture in the same way as for the Jacobian conjecture.

3.2 The Weyl algebra in finite characteristic as an Azumaya algebra

It is a classical fact that in finite characteristic the algebra $A_{n,R}$ has a big center, and it is moreover an Azumaya algebra of its center (see [8]).

We will use the following slightly non-standard definition of an Azumaya algebra (see e.g. [7], Proposition 2.1, Chapter IV):

Definition 1 *For a commutative ring R an Azumaya algebra over R of rank $N \geq 1$ is an associative unital algebra A over R which is a finitely generated R-module and such that there exists a finitely generated faithfully flat extension $R' \subset R$ of R such that the pullback algebra $A' := A \otimes_R R'$ is isomorphic to the matrix algebra*

$$\mathrm{Mat}(N \otimes N, R') = \mathrm{Mat}(N \times N, \mathbb{Z}) \otimes R'$$

as an algebra over R'.

It follows by descent that the center of an Azumaya algebra over R coincides with R. Also, an Azumaya algebra A considered as a R-module is a finitely generated projective module, in other words, a vector bundle over $\mathsf{Spec}\, R$. This bundle has rank N^2, its fibers are associative algebras, and the fiber over any point of $\mathsf{Spec}\, R$ over an algebraically closed field $\mathbf{k}$ is isomorphic to the matrix algebra $\mathrm{Mat}(N \times N, \mathbf{k})$.

Proposition 1 *For any commutative algebra R over $\mathbb{Z}/p\mathbb{Z}$ where p is a prime, the algebra $A_{n,R}$ is an Azumaya algebra of rank p^n over $R[x_1, \dots, x_{2n}]$. The central element of $A_{n,R}$ corresponding to variable x_i is y_i^p.*

Proof: Let us introduce a faithfully flat extension $R' := R[\xi_1, \dots, \xi_{2n}]$ of $R[x_1, \dots, x_{2n}]$, where the inclusion of $R[x_1, \dots, x_{2n}]$ into R' is given by

$$x_i \mapsto \xi^p, \quad i \in \{1, \dots, 2n\} .$$

We claim that the algebra over R'

$$A' := A_{n,R} \otimes_{R[x_1,\dots,x_{2n}]} R[\xi_1, \dots, \xi_{2n}]$$

is isomorphic to the matrix algebra of rank p^n over R'. Namely, the algebra A' considered as an algebra over $R' = R[\xi_1, \dots, \xi_{2n}]$, has generators y_i, $i \in \{1, \dots, 2n\}$ and defining relations

$$[y_i, y_j] = \omega_{ij}, \;\; y_i^p = \xi_i^p \; .$$

Introduce a new set of generators $y_i' \in A', \;\; i \in \{1, \dots, 2n\}$ by the formula

$$y_i' := y_i - \xi_i \; .$$

These generators have defining relations

$$[y_i', y_j'] = \omega_{ij}, \;\; (y_i')^p = 0 \; .$$

Hence, we see that the algebra A' over R' is isomorphic to the tensor product over $\mathbb{Z}/p\mathbb{Z}$ of R' and a finite-dimensional algebra over $\mathbb{Z}/p\mathbb{Z}$ given by the generators $(y_i')_{1 \le i \le 2n}$ and the relations as above. The last algebra is the tensor product of n copies of its version in the case $n = 1$. The statement of the proposition now follows from the following

Lemma 2 *For any prime number p the algebra A over $\mathbb{Z}/p\mathbb{Z}$ with two generators y_1, y_2 and relations*

$$[y_1, y_2] = 1, \, y_1^p = y_2^p = 0$$

is isomorphic to $\mathrm{Mat}(p \times p, \mathbb{Z}/p\mathbb{Z})$.

Proof: Consider the finite ring $B := \mathbb{Z}/p\mathbb{Z}[x]/(x^p) = \mathbb{Z}[x]/(x^p, p)$. It is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^p$ as an abelian group (and as $\mathbb{Z}/p\mathbb{Z}$-module). Differential operators Y_1, Y_2 acting on B and given by the formulas

$$Y_1(f) = df/dx, \;\; Y_2 f = xf, \;\; f \in B$$

are well-defined, and satisfy the relations $[Y_1, Y_2] = 1$, $Y_1^p = Y_2^p = 0$. Hence we obtain a homomorphism $A \to \mathrm{End}_{\mathbb{Z}/p\mathbb{Z}-mod}(B)$. A direct calculation shows that this is an isomorphism. $\square$

4 The proof of the implication $\mathrm{JC}_{2n} \to \mathrm{DC}_n$

Let us assume that the conjecture JC_{2n} (phrased in the form of Conjecture 1) is true, our goal is to prove DC_n.

Let $f : A_{n,\mathbb{C}} \to A_{n,\mathbb{C}}$ be an endomorphism of degree $\le d$. We have to prove that f is invertible.

Denote by R the subring of $\mathbb{C}$ generated by the coefficients of elements $f(y_i) \in A_{n,\mathbb{C}},\ , i \in \{1, \dots, 2n\}$ in the standard basis of $A_{n,\mathbb{C}}$. The ring R is a finitely generated integral domain. Moreover, we may assume that for any prime p the ring R/pR is either zero or an integral domain, in particular it has no non-zero nilpotents. In order to achieve this property it is enough to extend R by adding inverses to finitely many primes.

For any prime p the endomorphism f induces an endomorphism

$$f_p : A_{n,R/pR} \to A_{n,R/pR}$$

of an Azumaya algebra of rank p^n over

$$C_p := R/pR[x_1, \dots, x_{2n}] = \mathrm{Center}(A_{n,R/pR}) \ .$$

The following result was proved first by Y. Tsuchimoto [10], it follows also from a more general recent result from [2].

Proposition 2 *The endomorphism f_p maps C_p to itself.*

Proof: Denote by $\mathbf{k}$ an algebraically closed field of characteristic p. For any $\mathbf{k}$-point v of $\mathsf{Spec}\, C_p$ the fiber A_v is an algebra over $\mathbf{k}$ isomorphic to $\mathrm{Mat}(p^n \times p^n, \mathbf{k})$.

Lemma 3 *An element $a \in A_{n,R/pR}$ belongs to C_p if and only if for any $\mathbf{k}$-point v of $\mathsf{Spec}\, C_p$ where $\mathbf{k}$ is an algebraically closed field, the image of a in A_v is central, i.e. it is a scalar matrix.*

Proof: One direction is obvious, i.e. if a is central than its image in A_v is central. Conversely, if $a \in A_{n,R/pR}$ is not central then there exists $b \in A_{n,R/pR}$ such that $[a, b] \neq 0$. For any non-zero section s of the vector bundle $A_{n,R/pR}/C_p$ there exists a $\mathbf{k}$-point at which this section does not vanish, because the algebra $C_p = R/pR[x_1, \dots, x_{2n}]$ has no non-zero nilpotents by our assumption that R/pR has no non-zero nilpotents. We apply this

argument to the section $s = [a, b]$ and conclude that the image of a in A_v is not central for some v. $\square$

Let $a \in C_p \subset A_{n,R/pR}$ be a central element. We want to prove that $f(a)$ is central. Assume the opposite. Then by the above lemma there exists a homomorphism $\rho : A_{n,R/pR} \to \mathrm{Mat}(p^n \times p^n, \mathbf{k})$ such that $\rho(f(a))$ is not a scalar matrix. Let us denote by $V_0 \simeq \mathbf{k}^{p^n}$ the module over $A_{n,R/pR}$ associated to the homomorphism $f \circ \rho$. Our assumption mean that V_0 considered as a module over $\mathbf{k} \otimes C_p$ is *not* isomorphic to the sum of p^n copies of the simple module $M_v \simeq \mathbf{k}$ associated with any $\mathbf{k}$-point v of $\mathsf{Spec}\, C_p$. The support of the module V_0 is a non-empty finite subscheme of $\mathsf{Spec}\, C_p$ defined over $\mathbf{k}$, hence there exists a $\mathbf{k}$-point v in it support. Moreover, for any such point v the tensor product $V := V_0 \otimes_{C_v \otimes \mathbf{k}} M_v$ is a vector space over $\mathbf{k}$ such that $0 < \dim V < \dim V_0$. The algebra $A_{n,R/pR}$ maps to the algebra of endomorphisms of C_p-module V_0, hence it maps to the algebra of $\mathbf{k}$-linear endomorphisms of V. In this representation of $A_{n,R/pR}$ the center C_p acts by scalars, by the nature of the construction.

Therefore, we obtain a homomorphism $C_p \to \mathbf{k}$, i.e. a $\mathbf{k}$-point v of $\mathsf{Spec}\, C_p$, and a homomorphism of $\mathbf{k}$-algebras

$$A_v \simeq \mathrm{Mat}(p^n \times p^n, \mathbf{k}) \to \mathrm{Mat}(M \times M, \mathbf{k}), \;\; 0 < M < p^n ,$$

here $M := \dim V$. This is impossible because $\mathrm{Mat}(p^n \times p^n, \mathbf{k})$ is simple and $0 < \dim_{\mathbf{k}} \mathrm{Mat}(M \times M, \mathbf{k}) < \dim_{\mathbf{k}} \mathrm{Mat}(p^n \times p^n, \mathbf{k})$. We obtain a contradiction. The Proposition is proven. $\square$

Denote by f_p^{centr} the endomorphism of C_p induced by f. Our next goal is to prove that f_p^{centr} **preserves certain R/pR-linear Poisson bracket** on C_p.

Namely, we define an operation $\{\, ,\, \} : C_p \otimes_{R/pR} C_p \to C_p$ by the formula

$$\{a, b\} = \frac{[\tilde{a}, \tilde{b}]}{p} \pmod{pA_{n,R}} \in A_{n,R/pR} = A_{n,R}/pA_{n,R}$$

where $\tilde{a}, \tilde{b} \in A_{n,R}$ are arbitrary lifts of the elements $a, b \in C_p \subset A_{n,R/pR}$. First of all, it is easy to see that the commutator $[\tilde{a}, \tilde{b}]$ vanishes modulo p, hence the division by p makes sense. It is uniquely defined because R and hence $A_{n,R}$ both have no torsion. A straightforward check shows that $\{a, b\}$ defined as above does not depend on the choice of the lifts $\tilde{a}, \tilde{b}$, and it belongs to the center C_p. Moreover, the commutator $\{\, ,\, \}$ on C_p is a R/pR-linear, skew-symmetric operation satisfying the Jacobi identity (hence C_p becomes a Lie

algebra), and for any $a \in C_p$ the operator $\{a, \cdot\} : C_p \to C_p$ is a R/pR-linear derivation of C_p, i.e. the bracket satisfies the Leibniz rule

$$\{a, bb'\} = \{a, b\}b' + \{a, b'\}b \ .$$

Lemma 4 *The above defined canonical Poisson bracket on* $C_p \simeq R/p[x_1, \ldots, x_{2n}]$ *is given by the standard formula*

$$\{a, b\} = \sum_{i=1}^{n} \left(\frac{\partial a}{\partial x_i} \frac{\partial b}{\partial x_{n+i}} - \frac{\partial b}{\partial x_i} \frac{\partial a}{\partial x_{n+i}} \right) \ .$$

Proof: By the Leibniz rule it follows that it suffices to calculate the bracket $\{x_i, x_j\}$ for any two generators of C_p. The calculation reduces to the case $n = 1$. It is convenient to calculate first the commutator in the algebra $A_{1,\mathbb{Z}}$ and then make the reduction modulo p:

$$\frac{1}{p}[(d/dx)^p, x^p] = \frac{1}{p} \sum_{i=0}^{p-1} \frac{(p!)^2}{(i!)^2 (p-i)!} x^i (d/dx)^i = -1 \pmod p$$

Then the statement of the lemma follows immediately. □

The next lemma follows directly from the definition of the bracket:

Lemma 5 *The homomorphism* $f_p^{centr} : C_p \to C_p$ *preserves the canonical Poisson bracket.*

It is well-known in symplectic geometry that a non-degenerate Poisson structure on a C^∞ manifold X is essentially the same as a symplectic structure, i.e. a non-degenerate closed 2-form. The same is true in the algebraic context, in characteristic > 2. Namely, a Poisson bracket gives a section

$$\alpha \in \Gamma(\mathsf{Spec}\, C_p, \wedge^2 T_{\mathsf{Spec}\, C_p / \mathsf{Spec}\, R/pR})$$

of the wedge square of the tangent bundle, defined by the formula

$$\{f, g\} = \langle df \wedge dg, \alpha \rangle \in C_p, \ \ \forall f, g \in C_p \ .$$

This section can be interpreted as an operator from the cotangent bundle to the tangent bundle. This operator is invertible in our case, the inverse operator can be interpreted as a 2-form

$$\omega := \alpha^{-1} = \frac{1}{2} \sum_{1 \le i,j \le 2n} \omega_{ij} dx_i \wedge dx_j = \sum_{i=1}^{n} dx_i \wedge dx_{n+i} \ .$$

Lemma 6 *For $p > n$ the endomorphism f_p^{centr} of $C_p = R/pR[x_1, \ldots, x_{2n}]$ preserves the top-degree form $dx_1 \wedge \cdots \wedge dx_{2n} \in \Omega^{2n}(B/(R/pR))$.*

Proof: It follows from the previous lemma that f_{centr} preserves the symplectic 2-form ω. The volume form from above is equal to $\pm\omega^n/n!$ for $p > n$. $\square$

The next result implies that the degree of f_p^{centr} is $\leq d$.

Proposition 3 *For any field $\mathbf{k}$ of characteristic p and any $\mathbf{k}$-point v of $\mathsf{Spec}\, R$, the degree of f_v (as an endomorphism of the Weyl algebra $A_{n,\mathbf{k}}$) is equal to the degree of f_v^{centr} (as an endomorphism of the polynomial algebra) where f_v^{centr} is the endomorphism of $\mathrm{Center}(A_{n,\mathbf{k}}) \simeq \mathbf{k}[x_1, \ldots, x_{2n}]$ induced from f_p^{centr}.*

Proof: The degree of f_v^{centr} is defined as the maximum over $i \in \{1, \ldots, 2n\}$ of the degrees of polynomials $f_v^{centr}(x_i)$. The degree of endomorphism f_v is defined as the maximum over $i \in \{1, \ldots, 2n\}$ of the degrees (in the sense of Bernstein filtration, by the degree of monomials in the standard basis of $A_{n,\mathbf{k}}$) of elements $f_v(y_i)$. We claim that for each index i both degrees coincide with each other. The reason is the following. Let d_i be the degree of $f_v(y_i)$. We claim that the degree of $f_v(y_i^p) = (f_v(y_i))^p$ considered as an element of $A_{n,\mathbf{k}}$, is equal to pd_i. It follows from the following

Lemma 7 *The degree is an additive character of the multiplicative monoid of non-zero elements in $A_{n,\mathbf{k}}$.*

Proof: It follows immediately from the consideration of Bernstein filtration on $A_{n,\mathbf{k}}$ and the remark that the product of non-zero homogeneous polynomials is a non-zero polynomial. $\square$

The degree of $f_v(y_i^p)$ considered as an element of $\mathrm{Center}(A_{n,\mathbf{k}})$ is $1/p$ times its degree in $A_{n,\mathbf{k}}$, i.e. it is equal to d_i. Proposition 3 is proven. $\square$

Now we can use finally our main assumption that the Jacobian conjecture JC_{2n} holds. Namely, by its reformulation (in form of Conjecture 1), we conclude that there exists an integer $d' \geq 1$ and a finite set of primes S (the union of the set of excluded primes for JC_{2n} in form of Conjecture 1, and the set of primes $\leq n$), such that for any algebraically closed field $\mathbf{k}$ such that $p = \mathrm{char}\,(\mathbf{k}) \notin S \cup \{0\}$ and any $\mathbf{k}$-point v of $\mathsf{Spec}\, R$, the pullback f_v^{centr} to v of f_p^{centr} is invertible and the inverse endomorphism of $\mathbf{k}[x_1, \ldots, x_{2n}]$ has the degree $\leq d'$.

The following result is a particular case of a more general statement proven in [2], and also follows from [10].

Proposition 4 *For any v as above the endomorphism f_v of $A_{n,\mathbf{k}}$ is invertible.*

Proof: We may assume that $\mathbf{k}$ is algebraically closed. The endomorphism f_v of Azumaya algebra $A_{n,\mathbf{k}}$ preserves the center and is invertible on the center. Thus, it gives a C_v-linear homomorphism g_v from one Azumaya algebra of rank p^n over $C_v := \mathbf{k}[x_1, \dots, x_{2n}]$ (here we mean the algebra $A_{n,\mathbf{k}}$), to another Azumaya algebra of rank p^n (the pullback of $A_{n,\mathbf{k}}$ by f_v^{centr}). Any such a homomorphism restricts to an isomorphism after the reduction to any $\mathbf{k}$-point of C_v, because any homomorphim of associative $\mathbf{k}$-algebras

$$\mathrm{Mat}(N \times N, \mathbf{k}) \to \mathrm{Mat}(N \times N, \mathbf{k}),\ N := p^n$$

is an isomorphism. Therefore, g_v is an isomorphism of vector bundles. $\square$

Finally, the degree of the inverse to f_v is $\leq d'$, as follows directly from Proposition 3. The conclusion is that for any point v of $\mathsf{Spec}\, R$ over an field $\mathbf{k}$ of finite characteristic $p \notin S$, the corresponding point of the scheme of finite type $\mathrm{DE}_{n,d}$ (see Section 3.1 for the notation) belongs to the constructible set $\mathrm{DA}_{n,d,d'}$. This implies (see Lemma 1) that f is invertible after the localization to zero characteristic, and the inverse endomorphism has degree $\leq d'$. Theorem 1 is proven. $\square$

Remark 4 *It is interesting that Poisson brackets appear in another situation related to polynomial automorphisms. The Poisson algebra structure was used by I. Shestakov and U. Umirbaev in their proof that the Nagata automorphism is wild (see [9]).*

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